

Transverse conformal Killing forms on Kähler foliations

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Abstract. On a closed, connected Riemannian manifold with a Kähler foliation of codimension $q = 2m$, any transverse Killing r (≥ 2)-form is parallel (S. D. Jung and M. J. Jung [4], Bull. Korean Math. Soc. 49 (2012)). In this paper, we study transverse conformal Killing forms on Kähler foliations and prove that if the foliation is minimal, then for any transversal conformal Killing r -form ϕ ($2 \leq r \leq q - 2$), $J\phi$ is parallel. Here J is defined in Section 4.

1 Introduction

On Riemannian manifolds, conformal Killing forms are generalizations of conformal Killing fields, which were introduced by K. Yano [20] and T. Kashiwada [10,11]. Many researchers have studied the conformal Killing forms [13, 16, 17, 18]. On a foliated Riemannian manifold, we can study the analogous problems. Let \mathcal{F} be a transversally oriented Riemannian foliation on a compact oriented Riemannian manifold M with codimension q . A transversal conformal Killing field is a normal field with a flow preserving the conformal class of the transverse metric. As a generalization of a transversal conformal Killing field, we define the *transverse conformal Killing r -forms* ϕ as follows: for any vector field X normal to the foliation,

$$\nabla_X \phi - \frac{1}{r+1} i(X) d\phi + \frac{1}{q-r+1} X^\flat \wedge \delta_T \phi = 0,$$

where r is the degree of the form ϕ and X^\flat is the dual 1-form of X . For the definition of δ_T , see Section 3. The transverse conformal Killing forms ϕ with $\delta_T \phi = 0$ are called *transverse Killing forms*. Recently, S. D. Jung and K. Richardson [6] studied the transverse Killing and conformal Killing forms on Riemannian foliations. And S. D. Jung and M. J. Jung [4] studied some properties of the transverse

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Killing forms on Kähler foliations. That is, on a closed, connected Riemannian manifold with a Kähler foliation of codimension $q = 2m$, any transverse Killing $r(\geq 2)$ -form is parallel. In this paper, we study the transverse conformal Killing forms on Kähler foliations. In section 2, we review the basic facts on a Riemannian foliation. In section 3, we study the transverse conformal Killing forms and curvature properties on Riemannian foliations. In section 4, we study the curvatures and several operators on Kähler foliations. In section 5, we prove the following: on a Kähler foliation with $q = 2m$, if ϕ is a transverse conformal Killing m -form, then $J\phi$ is parallel. In particular, when (\mathcal{F}, J) is minimal, for any transverse conformal Killing r ($2 \leq r \leq q - 2$)-forms ϕ , $J\phi$ is also parallel. Here J is an extension of the complex structure J to the basic forms.

2 Preliminaries

Let (M, g_M, \mathcal{F}) be a $(p + q)$ -dimensional Riemannian manifold with a foliation \mathcal{F} of codimension q and a bundle-like metric g_M with respect to \mathcal{F} . Then there exists an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0, \quad (2.1)$$

where L is the tangent bundle and $Q = TM/L$ is the normal bundle of \mathcal{F} . The metric g_M determines an orthogonal decomposition $TM = L \oplus L^\perp$, identifying Q with L^\perp and inducing a metric g_Q on Q . Let ∇ be the transverse Levi-Civita connection on Q , which is torsion-free and metric with respect to g_Q [7]. Let $R^\nabla, K^\nabla, \rho^\nabla$ and σ^∇ be the transversal curvature tensor, transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to ∇ , respectively. Let $\Omega_B^*(\mathcal{F})$ be the space of all *basic forms* on M , i.e.,

$$\Omega_B^*(\mathcal{F}) = \{\phi \in \Omega^*(M) \mid i(X)\phi = 0, \ i(X)d\phi = 0, \ \forall X \in \Gamma L\}. \quad (2.2)$$

Then $L^2\Omega^*(M)$ is decomposed as [1]

$$L^2\Omega(M) = L^2\Omega_B(\mathcal{F}) \oplus L^2\Omega_B(\mathcal{F})^\perp. \quad (2.3)$$

Now we define the connection ∇ on $\Omega_B^*(\mathcal{F})$, which is induced from the connection ∇ on Q and Riemannian connection $\overset{\circ}{\nabla}^M$ of g_M . This connection ∇ extends the partial Bott connection $\overset{\circ}{\nabla}$ given by $\overset{\circ}{\nabla}_X\phi = \theta(X)\phi$ for any $X \in \Gamma L$ [9], where $\theta(X)$ is the transversal Lie derivative. Then the basic forms are characterized by $\Omega_B^*(\mathcal{F}) = \text{Ker} \overset{\circ}{\nabla} \subset \Gamma(\wedge Q^*(\mathcal{F}))$. By a direct calculation, we have the following lemma.

Lemma 2.1 *Let (M, g_M, \mathcal{F}) be a Riemannian manifold with a foliation \mathcal{F} and a bundle-like metric g_M . Then for any $X, Y, Z \in \Gamma Q$,*

$$[R^\nabla(X, Y), i(Z)] = i(R^\nabla(X, Y)Z).$$

The exterior differential d on the de Rham complex $\Omega^*(M)$ restricts a differential $d_B : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r+1}(\mathcal{F})$. Let $\kappa \in Q^*$ be the mean curvature form of \mathcal{F} . Then it is well known that the basic part κ_B of κ is closed [1]. We now recall the star operator $\bar{*} : \Omega^r(M) \rightarrow \Omega^{q-r}(M)$ given by [15, 19]

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \quad \forall \phi \in \Omega^r(M), \quad (2.4)$$

where $\chi_{\mathcal{F}}$ is the characteristic form of \mathcal{F} and $*$ is the Hodge star operator associated to g_M . The operator $\bar{*}$ maps basic forms to basic forms. For any $\phi, \psi \in \Omega_B^r(\mathcal{F})$, $\phi \wedge \bar{*}\psi = \psi \wedge \bar{*}\phi$ and also $\bar{*}^2\phi = (-1)^{r(q-r)}\phi$ [15]. Let ν be the transversal volume form, i.e., $*\nu = \chi_{\mathcal{F}}$. The pointwise inner product $\langle \cdot, \cdot \rangle$ on $\Lambda^r Q^*$ is defined uniquely by

$$\langle \phi, \psi \rangle \nu = \phi \wedge \bar{*}\psi. \quad (2.5)$$

The global inner product $(\cdot, \cdot)_B$ on $L^2\Omega_B^r(\mathcal{F})$ is defined by

$$(\phi, \psi)_B = \int_M \langle \phi, \psi \rangle \mu_M, \quad \forall \phi, \psi \in \Omega_B^r(\mathcal{F}), \quad (2.6)$$

where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form with respect to g_M . With respect to this scalar product, the formal adjoint $\delta_B : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r-1}(\mathcal{F})$ of d_B is given by [15]

$$\delta_B\phi = (-1)^{q(r+1)+1} \bar{*}d_T \bar{*}\phi = \delta_T\phi + i(\kappa_B^\sharp)\phi, \quad (2.7)$$

where $d_T = d - \kappa_B \wedge$ and $\delta_T = (-1)^{q(r+1)+1} \bar{*}d\bar{*}$ is the formal adjoint operator of d_T . Here $(\cdot)^\sharp$ is a g_Q -dual vector to (\cdot) . The basic Laplacian Δ_B is given by $\Delta_B = d_B\delta_B + \delta_B d_B$. Let $\{E_a\}(a = 1, \dots, q)$ be a local orthonormal basic frame on Q . We define $\nabla_{\text{tr}}^* \nabla_{\text{tr}} : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^r(\mathcal{F})$ by

$$\nabla_{\text{tr}}^* \nabla_{\text{tr}}\phi = - \sum_a \nabla_{E_a, E_a}^2 \phi + \nabla_{\kappa_B^\sharp} \phi, \quad \phi \in \Omega_B^r(\mathcal{F}), \quad (2.8)$$

where $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for any $X, Y \in TM$. Then the operator $\nabla_{\text{tr}}^* \nabla_{\text{tr}}$ is positive definite and formally self adjoint on the space of basic forms [2]. We define the bundle map $A_Y : \Lambda^r Q^* \rightarrow \Lambda^r Q^*$ for any $Y \in TM$ [8] by

$$A_Y\phi = \theta(Y)\phi - \nabla_Y\phi. \quad (2.9)$$

For any $X \in \Gamma L$, $\theta(X)\phi = \nabla_X \phi$ [9] and so $A_X \phi = 0$. Now we define the curvature endomorphism $F : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^r(\mathcal{F})$ by

$$F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b) R^\nabla(E_b, E_a) \phi, \quad (2.10)$$

where θ^a is a g_Q -dual 1-form to E_a . Then we have the generalized Weitzenböck formula.

Theorem 2.2 [3] *On a Riemannian foliation \mathcal{F} , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\Delta_B \phi = \nabla_{\text{tr}}^* \nabla_{\text{tr}} \phi + F(\phi) + A_{\kappa_B^\sharp} \phi.$$

In particular, if ϕ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

Corollary 2.3 *On a Riemannian foliation \mathcal{F} , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\frac{1}{2} \Delta_B |\phi|^2 = \langle \Delta_B \phi, \phi \rangle - |\nabla_{\text{tr}} \phi|^2 - \langle F(\phi), \phi \rangle - \langle A_{\kappa_B^\sharp} \phi, \phi \rangle.$$

Now, we recall the following generalized maximum principle.

Theorem 2.4 [12] *Let \mathcal{F} be a Riemannian foliation on a closed, connected Riemannian manifold (M, g_M) . If $(\Delta_B - \kappa_B^\sharp) f \geq 0$ (or ≤ 0) for any basic function f , then f is constant.*

3 The transverse conformal Killing forms

Let (M, g_M, \mathcal{F}) be a Riemannian manifold with a foliation \mathcal{F} of codimension q and a bundle-like metric g_M .

Definition 3.1 A basic r -form $\phi \in \Omega_B^r(\mathcal{F})$ is called a *transverse conformal Killing r -form* if for any vector field $X \in \Gamma Q$,

$$\nabla_X \phi = \frac{1}{r+1} i(X) d_B \phi - \frac{1}{r^*+1} X^\flat \wedge \delta_T \phi,$$

where $r^* = q - r$ and X^\flat is the g_Q -dual 1-form of X . In addition, if the basic r -form ϕ satisfies $\delta_T \phi = 0$, it is called a *transverse Killing r -form*.

Note that a transverse conformal Killing 1-form (resp. transverse Killing 1-form) is a g_Q -dual form of a transversal conformal Killing field (resp. transversal Killing field).

Proposition 3.2 [6] *Let ϕ be a transverse conformal Killing r -form. Then*

$$F(\phi) = \frac{r}{r+1} \delta_T d_B \phi + \frac{r^*}{r^*+1} d_B \delta_T \phi, \quad (3.1)$$

$$\nabla_{\text{tr}}^* \nabla_{\text{tr}} \phi = \frac{1}{r+1} \delta_B d_B \phi + \frac{1}{r^*} d_T \delta_T \phi. \quad (3.2)$$

Lemma 3.3 [6] *Let ϕ be a transverse conformal Killing r -form. Then*

$$\begin{aligned} \nabla_X \nabla_Y \phi &= \frac{1}{r+1} \{i(\nabla_X Y) d_B \phi + i(Y) \nabla_X d_B \phi\} \\ &\quad - \frac{1}{r^*+1} \{\nabla_X Y^{\flat} \wedge \delta_T \phi + Y^{\flat} \wedge \nabla_X \delta_T \phi\} \end{aligned}$$

for any $X, Y \in \Gamma Q$.

We define the operators $R_{\pm}^{\nabla}(X) : \wedge^r Q^* \rightarrow \wedge^{r \pm 1} Q^*$ for any $X \in TM$ by

$$R_+^{\nabla}(X) \phi = \sum_a \theta^a \wedge R^{\nabla}(X, E_a) \phi, \quad (3.3)$$

$$R_-^{\nabla}(X) \phi = \sum_a i(E_a) R^{\nabla}(X, E_a) \phi. \quad (3.4)$$

Then we have the following lemma.

Lemma 3.4 *Let ϕ be a transverse conformal Killing r -form. Then for all $X \in \Gamma Q$,*

$$\nabla_X d_B \phi = \frac{r+1}{r} \{R_+^{\nabla}(X) \phi + \frac{1}{r^*+1} X^{\flat} \wedge d_B \delta_T \phi\}, \quad (3.5)$$

$$\nabla_X \delta_T \phi = -\frac{r^*+1}{r^*} \{R_-^{\nabla}(X) \phi + \frac{1}{r+1} i(X) \delta_T d_B \phi\}. \quad (3.6)$$

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$. Since $\sum_a \theta^a \wedge i(E_a) \phi = r \phi$ for any $\phi \in \Omega_B^r(\mathcal{F})$, from Lemma 3.3

$$R_+^{\nabla}(X) \phi = \frac{r}{r+1} \nabla_X d_B \phi - \frac{1}{r^*+1} X^{\flat} \wedge d_B \delta_T \phi,$$

which proves (3.5). The proof of (3.6) is similar. \square

Proposition 3.5 *Let ϕ be a transverse conformal Killing r -form. Then for any $X, Y \in \Gamma Q$,*

$$\begin{aligned} R^\nabla(X, Y)\phi &= \frac{1}{rr^*} \left(Y^\flat \wedge i(X) - X^\flat \wedge i(Y) \right) F(\phi) \\ &+ \frac{1}{r} \left(i(Y)R_+^\nabla(X) - i(X)R_+^\nabla(Y) \right) \phi + \frac{1}{r^*} \left(Y^\flat \wedge R_-^\nabla(X) - X^\flat \wedge R_-^\nabla(Y) \right) \phi. \end{aligned}$$

Proof. Let ϕ be the transverse conformal Killing r -form. From Lemma 3.3,

$$\begin{aligned} R^\nabla(X, Y)\phi &= \frac{1}{r+1} \{ i(Y)\nabla_X d_B \phi - i(X)\nabla_Y d_B \phi \} \\ &- \frac{1}{r^*+1} \{ Y^\flat \wedge \nabla_X \delta_T \phi - X^\flat \wedge \nabla_Y \delta_T \phi \}. \end{aligned}$$

From Lemma 3.4, we have

$$\begin{aligned} R^\nabla(X, Y)\phi &= \frac{1}{r} \{ i(Y)R_+^\nabla(X) - i(X)R_+^\nabla(Y) \} \phi + \frac{1}{r^*} \{ Y^\flat \wedge R_-^\nabla(X) - X^\flat \wedge R_-^\nabla(Y) \} \phi \\ &- \left(X^\flat \wedge i(Y) - Y^\flat \wedge i(X) \right) \left\{ \frac{1}{r(r^*+1)} d_B \delta_T \phi + \frac{1}{r^*(r+1)} \delta_T d_B \phi \right\}. \end{aligned}$$

Hence the proof follows from (3.1). \square

Lemma 3.6 *Let ϕ be a transverse conformal Killing r -form. Then*

$$\sum_a i(E_a)R_-^\nabla(E_a)\phi = \sum_a \theta^a \wedge R_+^\nabla(E_a)\phi = 0.$$

Proof. Since ϕ is a transverse conformal Killing r -form, from Proposition 3.5,

$$\begin{aligned} \sum_a i(E_a)R_-^\nabla(E_a)\phi &= \frac{2}{r^*} \sum_{a,b} i(E_a)i(E_b) \{ \theta^b \wedge R_-^\nabla(E_a)\phi \} \\ &= \frac{2(r^*+1)}{r^*} \sum_a i(E_a)R_-^\nabla(E_a)\phi, \end{aligned}$$

which means that $\sum_a i(E_a)R_-^\nabla(E_a)\phi = 0$. Similarly, we have

$$\begin{aligned} \sum_a \theta^a \wedge R_+^\nabla(E_a)\phi &= \frac{2}{r} \sum_{a,b} \theta^a \wedge \theta^b \wedge \{ i(E_b)R_+^\nabla(E_a)\phi \} \\ &= \frac{2(r+1)}{r} \sum_a \theta^a \wedge R_+^\nabla(E_a)\phi, \end{aligned}$$

which proves the second equality. \square

4 Curvatures on a Kähler foliation

Let (M, g_M, J, \mathcal{F}) be a compact Riemannian manifold with a Kähler foliation \mathcal{F} of codimension $q = 2m$ and a bundle-like metric g_M [14]. Note that for any $X, Y \in \Gamma Q$,

$$\Omega(X, Y) = g_Q(X, JY) \quad (4.1)$$

defines a basic 2-form Ω , which is closed as consequence of $\nabla g_Q = 0$ and $\nabla J = 0$, where $J : Q \rightarrow Q$ is an almost complex structure on Q . Then

$$\Omega = -\frac{1}{2} \sum_{a=1}^{2m} \theta^a \wedge J\theta^a. \quad (4.2)$$

Moreover, we have the following identities: for any $X, Y \in \Gamma Q$,

$$R^\nabla(X, Y)J = JR^\nabla(X, Y), \quad R^\nabla(JX, JY) = R^\nabla(X, Y). \quad (4.3)$$

Trivially, we have the following lemma.

Lemma 4.1 *On a Kähler foliation (\mathcal{F}, J) , the following holds:*

$$\sum_a \theta^a \wedge \rho^\nabla(E_a)^\flat = 0.$$

Proof. By a direct calculation, we have

$$\begin{aligned} \sum_a \theta^a \wedge \rho^\nabla(E_a)^\flat &= \sum_{a,b} \theta^a \wedge R^\nabla(E_a, JE_b)J\theta^b \\ &= \sum_{a,b,c} \theta^a \wedge g_Q(R^\nabla(E_a, JE_b)JE_b, E_c)\theta^c \\ &= \sum_{a,b} R^\nabla(E_b, JE_a)J\theta^b \wedge \theta^a \\ &= \sum_a \rho^\nabla(E_a)^\flat \wedge \theta^a, \end{aligned}$$

which proves (4.4). \square

Lemma 4.2 *On a Kähler foliation (\mathcal{F}, J) , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\sum_a i(E_a) R_+^\nabla(E_a) \phi = - \sum_a \theta^a \wedge R_-^\nabla(E_a) \phi = -F(\phi), \quad (4.4)$$

$$\sum_a i(E_a) R_-^\nabla(JE_a) \phi = \sum_a \theta^a \wedge R_+^\nabla(JE_a) \phi = 0. \quad (4.5)$$

Proof. The proof of (4.5) is trivial. Note that for any $X, Y \in \Gamma Q$,

$$R^\nabla(JX, Y) = R^\nabla(JY, X). \quad (4.6)$$

From (4.7), the proof of (4.6) is trivial. \square

Lemma 4.3 *On a Kähler foliation (\mathcal{F}, J) , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\sum_a R_+^\nabla(JE_a) i(E_a) \phi = 0.$$

Proof. Let $\phi = \frac{1}{r!} \sum_{i_1, \dots, i_r} \phi_{i_1 \dots i_r} \theta^{i_1} \wedge \dots \wedge \theta^{i_r}$ be a basic r -form. Then by a long calculation, we have

$$\begin{aligned} & \sum_{a,b} \theta^a \wedge R^\nabla(JE_a, E_b) i(E_b) \phi \\ &= \frac{1}{r!} \sum_{i_1, \dots, i_r} \sum_{a, k < l} (-1)^{k+l-1} \phi_{i_1 \dots i_r} \theta^a \wedge \{R^\nabla(JE_a, E_{i_k}) \theta^{i_l} - R^\nabla(JE_a, E_{i_l}) \theta^{i_k}\} \wedge \psi_{k,l} \\ &= \frac{2}{r!} \sum_{i_1, \dots, i_r} \sum_{a, k < l} (-1)^{k+l-1} \phi_{i_1 \dots i_r} \theta^a \wedge R^\nabla(JE_a, E_{i_k}) \theta^{i_l} \wedge \psi_{k,l}, \end{aligned}$$

where $\psi_{k,l} = \theta^{i_1} \wedge \dots \wedge \hat{\theta}^{i_k} \wedge \dots \wedge \hat{\theta}^{i_l} \wedge \dots \wedge \theta^{i_r}$. From (4.7),

$$\sum_{i_k, i_l} \phi_{i_1 \dots i_r} R^\nabla(JE_{i_k}, E_{i_l}) = 0.$$

Hence, by the first Bianchi identity, we have

$$\begin{aligned} \sum_{a, i_k, i_l} \phi_{i_1 \dots i_r} \theta^a \wedge R^\nabla(JE_a, E_{i_k}) \theta^{i_l} &= \sum_{a, b, i_k, i_l} \phi_{i_1 \dots i_r} g_Q(R^\nabla(JE_a, E_{i_k}) E_{i_l}, E_b) \theta^a \wedge \theta^b \\ &= \sum_{a, i_k, i_l} \phi_{i_1 \dots i_r} R^\nabla(E_{i_l}, E_a) J \theta^{i_k} \wedge \theta^a \\ &= \sum_{a, i_k, i_l} \phi_{i_1 \dots i_r} R^\nabla(JE_a, E_{i_k}) \theta^{i_l} \wedge \theta^a \\ &= \sum_{a, i_k, i_l} \phi_{i_1 \dots i_r} R^\nabla(JE_a, E_{i_k}) \theta^{i_l} \wedge \theta^a, \end{aligned}$$

which means

$$\sum_{a, i_1, \dots, i_r} \phi_{i_1 \dots i_r} \theta^a \wedge R^\nabla(JE_a, E_{i_k}) \theta^{i_l} = 0.$$

Hence the proof is completed. \square

Let $L : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r+2}(\mathcal{F})$ and $\Lambda : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r-2}(\mathcal{F})$ be given respectively by [5]

$$L(\phi) = \epsilon(\Omega)\phi, \quad \Lambda(\phi) = i(\Omega)\phi, \quad (4.7)$$

where $\epsilon(\Omega)\phi = \Omega \wedge \phi$ and $i(\Omega) = -\frac{1}{2} \sum_{a=1}^{2m} i(JE_a)i(E_a)$. Trivially, for any basic forms $\phi \in \Omega_B^r(\mathcal{F})$ and $\psi \in \Omega_B^{r+2}(\mathcal{F})$, $\langle L(\phi), \psi \rangle = \langle \phi, \Lambda(\psi) \rangle$. Moreover, for any basic r -form ϕ , $[\Lambda, L]\phi = \frac{1}{2}(q-2r)\phi$. Also, we have the following lemma.

Lemma 4.4 [5] *On a Kähler foliation (\mathcal{F}, J) , we have that for any $X \in \mathcal{Q}$,*

$$[L, i(X)] = \epsilon(JX^\flat), \quad [L, \epsilon(X^\flat)] = [\Lambda, i(X)] = 0, \quad [\Lambda, \epsilon(X^\flat)] = -i(JX),$$

where $\epsilon(\omega)\phi = \omega \wedge \phi$ for any $\omega \in \Omega_B^1(\mathcal{F})$.

Now, we define the operators $\tilde{J} : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^r(\mathcal{F})$ and $S : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^r(\mathcal{F})$ respectively by

$$\tilde{J}(\phi) = \sum_{a=1}^{2m} J\theta^a \wedge i(E_a)\phi, \quad (4.8)$$

$$S(\phi) = \sum_{a=1}^{2m} J\theta^a \wedge i(\rho^\nabla(E_a))\phi. \quad (4.9)$$

Trivially, if $\phi \in \Omega_B^1(\mathcal{F})$, then $\tilde{J}\phi = J\phi$. From now on, if we have no confusion, we write $\tilde{J} \equiv J$.

Lemma 4.5 *On a Kähler foliation (\mathcal{F}, J) , we have that for any $X, Y \in \mathcal{Q}$,*

$$[J, i(X)] = i(JX), \quad [J, \epsilon(X^\flat)] = \epsilon(JX^\flat), \quad [R^\nabla(X, Y), J] = 0.$$

Proof. The first two equations are trivial. Since $\sum_a R^\nabla(X, Y)J\theta^a \wedge i(E_a) + J\theta^a \wedge i(R^\nabla(X, Y)E_a) = 0$, for any $X, Y \in \mathcal{Q}$,

$$\begin{aligned} R^\nabla(X, Y)J\phi &= \sum_a J\theta^a \wedge i(E_a)R^\nabla(X, Y)\phi \\ &= JR^\nabla(X, Y)\phi, \end{aligned}$$

which proves the third equation. \square

Lemma 4.6 *On a Kähler foliation (\mathcal{F}, J) , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\sum_a R^\nabla(E_a, JE_a)\phi = -2S(\phi), \quad (4.10)$$

$$\sum_a \theta^a \wedge R_-^\nabla(JE_a)\phi = \sum_a i(E_a)R_+^\nabla(JE_a)\phi = S(\phi). \quad (4.11)$$

Proof. Note that for any $X \in \Gamma Q$,

$$\sum_a R^\nabla(E_a, JE_a)X^\flat = -2\rho^\nabla(JX)^\flat. \quad (4.12)$$

Let $\phi = \frac{1}{r!} \sum_{i_1, \dots, i_r} \phi_{i_1 \dots i_r} \theta^{i_1} \wedge \dots \wedge \theta^{i_r}$. From (4.12), we have

$$\begin{aligned} \sum_a R^\nabla(E_a, JE_a)\phi &= -\frac{2}{r!} \sum_{k, i_1, \dots, i_r} \phi_{i_1 \dots i_r} \theta^{i_1} \wedge \dots \wedge \rho^\nabla(JE_{i_k})^\flat \wedge \dots \wedge \theta^{i_r} \\ &= 2 \sum_a \theta^a \wedge i(\rho^\nabla(JE_a))\phi = -2S(\phi), \end{aligned}$$

which proves (4.10). From Lemma 2.1, we have

$$\sum_a R_+^\nabla(JE_a)i(E_a)\phi = \sum_a \theta^a \wedge R_-^\nabla(JE_a)\phi + \sum_a \theta^a \wedge i(\rho^\nabla(JE_a))\phi. \quad (4.13)$$

From Lemma 4.3 and (4.13), we have

$$\sum_a \theta^a \wedge R_-^\nabla(JE_a)\phi = S(\phi).$$

Moreover, since $\sum_a R^\nabla(JE_a, E_a)\phi = \sum_a \theta^a \wedge R_-^\nabla(JE_a)\phi + \sum_a i(E_a)R_+^\nabla(JE_a)\phi$, the proof of (4.11) follows. \square

Lemma 4.7 *On a Kähler foliation (\mathcal{F}, J) , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\sum_a \theta^a \wedge JR_-^\nabla(E_a)\phi = S(\phi) + F(J\phi), \quad (4.14)$$

$$\sum_a i(E_a)JR_+^\nabla(E_a)\phi = S(\phi) - F(J\phi). \quad (4.15)$$

Proof. From Lemma 4.5, we have

$$\sum_a \theta^a \wedge JR_-^\nabla(E_a)\phi = \sum_a J\{\theta^a \wedge R_-^\nabla(E_a)\phi\} - \sum_a J\theta^a \wedge R_-^\nabla(E_a)\phi.$$

From Lemma 4.2 and Lemma 4.6, (4.14) is proved. The proof of (4.15) is similar. \square

Lemma 4.8 *On a Kähler foliation (\mathcal{F}, J) , we have*

$$[J, L] = [J, \Lambda] = [F, J] = [F, \Lambda] = [S, J] = [S, \Lambda] = [S, L] = 0.$$

Proof. From Lemma 4.5, we have

$$\begin{aligned} [F, J]\phi &= - \sum_{a,b} J\theta^b \wedge i(E_a)R^\nabla(E_a, E_b)\phi - \sum_{a,b} \theta^b \wedge i(JE_a)R^\nabla(E_a, E_b)\phi \\ &= 0. \end{aligned}$$

Others are easily proved. \square

Now, we recall the operators $d_B^c : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r+1}(\mathcal{F})$ and $\delta_B^c : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r-1}(\mathcal{F})$, which are given by [5]

$$d_B^c \phi = \sum_{a=1}^{2n} J\theta^a \wedge \nabla_{E_a} \phi, \quad (4.16)$$

$$\delta_B^c \phi = \delta_T^c \phi + i(J\kappa_B^\#)\phi, \quad (4.17)$$

where $\delta_T^c \phi = - \sum_{a=1}^{2m} i(JE_a)\nabla_{E_a} \phi$. Trivially, δ_B^c is a formal adjoint of d_B^c and $\delta_B^{c^2} = d_B^{c^2} = 0$ [5]. Then we have the following lemma.

Lemma 4.9 [5] *On a Kähler foliation (\mathcal{F}, J) , we have that for any $X \in Q$,*

$$[L, d_B] = [L, d_B^c] = 0, \quad [L, \delta_B] = -d_T^c, \quad [L, \delta_B^c] = d_T, \quad (4.18)$$

$$[\Lambda, \delta_B] = [\Lambda, \delta_B^c] = 0, \quad [\Lambda, d_B] = \delta_T^c, \quad [\Lambda, d_B^c] = -\delta_T, \quad (4.19)$$

$$[J, d_B] = d_B^c, \quad [J, \delta_B] = \delta_B^c, \quad [J, d_B^c] = -d_B, \quad [J, \delta_B^c] = -\delta_B. \quad (4.20)$$

where $d_T^c = d_B^c - \epsilon(J\kappa_B)$.

Proof. Note that on Kähler foliations, $\nabla J = 0$ and then $\nabla \tilde{J} = 0$. Hence by Lemma 4.5, the proof follows. \square

Proposition 4.10 *On a Kähler foliation (\mathcal{F}, J) , we have*

$$d_T^c \delta_B + \delta_B d_T^c = d_B \delta_T^c + \delta_T^c d_B = 0, \quad (4.21)$$

$$d_B^c \delta_T + \delta_T d_B^c = d_T \delta_B^c + \delta_B^c d_T = 0, \quad (4.22)$$

$$\delta_B \delta_B^c + \delta_B^c \delta_B = d_B d_B^c + d_B^c d_B = 0. \quad (4.23)$$

Proof. From Lemma 4.9, we have

$$d_T^c \delta_B + \delta_B d_T^c = -[L, \delta_B] \delta_B - \delta_B [L, \delta_B] = 0.$$

Others are similary proved. \square

Now, we put that for any $X \in TM$,

$$e(X)\phi = \delta_B i(X)\phi + i(X)\delta_B \phi. \quad (4.24)$$

Then we have the following.

Lemma 4.11 *On a Kähler foliation (\mathcal{F}, J) , we have that for any $\phi \in \Omega_B^r(\mathcal{F})$,*

$$\begin{aligned} [J, \Delta_B] &= \{\theta(J\kappa_B^\sharp) + \theta(J\kappa_B^\sharp)^t\}, \\ [\Lambda, \Delta_B] &= e(J\kappa_B^\sharp), \end{aligned}$$

where $\theta(X)^t$ is a formal adjoint of $\theta(X)$ for any $X \in Q$.

Corollary 4.12 *On a minimal Kähler foliation (\mathcal{F}, J) , we have*

$$[J, \Delta_B] = [\Lambda, \Delta_B] = 0. \quad (4.25)$$

5 Transverse conformal Killing forms on Kähler foliations

Let (M, g_M, J, \mathcal{F}) be a compact Riemannian manifold with a Kähler foliation \mathcal{F} of codimension $q = 2m$ and a bundle-like metric g_M with respect to \mathcal{F} .

Proposition 5.1 *On a Kähler foliation (\mathcal{F}, J) , if ϕ is a transverse conformal Killing r -form, then*

$$(q + r^2 - qr)S(\phi) = F(J\phi). \quad (5.1)$$

Proof. Let ϕ be a transversal conformal Killing r -form. From Proposition 3.5,

$$\begin{aligned} \sum_a R^\nabla(E_a, JE_a)\phi &= \frac{2}{rr^*} JF(\phi) + \frac{2}{r} \sum_a i(JE_a)R_+^\nabla(E_a)\phi \\ &\quad + \frac{2}{r^*} \sum_a J\theta^a \wedge R_-^\nabla(E_a)\phi. \end{aligned}$$

Hence the proof follows from Lemma 4.6. \square

On the other hand, we also have the following proposition.

Proposition 5.2 *On a Kähler foliation (\mathcal{F}, J) , if ϕ is a transverse conformal Killing r -form, then*

$$(qr - q - r^2)S(\phi) = (r - 1)F(J\phi). \quad (5.2)$$

Proof. Let ϕ be a transverse conformal Killing r -form. Then from Lemma 4.6, we have

$$\sum_a \theta^a \wedge JR_-^\nabla(E_a)\phi = \sum_a \theta^a \wedge i(E_b)JR^\nabla(E_a, E_b)\phi + S(\phi). \quad (5.3)$$

Since $\sum_a \theta^a \wedge i(E_a)\phi = r\phi$ for any $\phi \in \Omega_B^r(\mathcal{F})$, by Proposition 3.5 and Lemma 4.5, we have

$$\begin{aligned} & \sum_b i(E_b)JR^\nabla(E_a, E_b)\phi \\ &= \frac{1}{rr^*} \left\{ (r^* - 1)Ji(E_a)F(\phi) - i(JE_a)F(\phi) + 2\theta^a \wedge F(\Lambda\phi) \right\} \\ &+ \frac{1}{r} \left\{ 2\Lambda R_+^\nabla(E_a)\phi - i(JE_a)F(\phi) + \sum_b i(E_a)i(E_b)JR_+^\nabla(E_b)\phi \right\} \\ &+ \frac{1}{r^*} \left\{ (r^* - 1)JR_-^\nabla(E_a)\phi - R_-^\nabla(JE_a)\phi + \theta^a \wedge i(E_b)JR_-^\nabla(E_b)\phi \right\}. \end{aligned}$$

From Lemma 3.6, Lemma 4.4 and Lemma 4.6, $\sum_a \theta^a \wedge \Lambda R_+^\nabla(E_a)\phi = -S(\phi)$. Hence, from Lemma 4.2 and Lemma 4.4 \sim Lemma 4.7, we have

$$\begin{aligned} & \sum_{a,b} \theta^a \wedge i(E_b)JR^\nabla(E_a, E_b)\phi \\ &= \frac{1}{rr^*} (rr^* - r - r^* + 2)F(J\phi) \\ &+ \frac{1}{r} \left\{ r \sum_a i(E_a)JR_+^\nabla(E_a)\phi + 2 \sum_a \theta^a \wedge \Lambda R_+^\nabla(E_a)\phi + F(J\phi) \right\} \\ &+ \frac{1}{r^*} \left\{ (r^* - 1) \sum_a \theta^a \wedge JR_-^\nabla(E_a)\phi - \sum_a \theta^a \wedge R_-^\nabla(JE_a)\phi \right\} \\ &= \frac{2-r}{rr^*} F(J\phi) + \frac{rr^* - 2r^* - r}{rr^*} S(\phi) + \frac{r^* - 1}{r^*} \sum_a \theta^a \wedge JR_-^\nabla(E_a)\phi. \end{aligned}$$

From (5.3), we have

$$\sum_a \theta^a \wedge JR_-^\nabla(E_a)\phi = \frac{2-r}{r} F(J\phi) + \frac{2rr^* - 2r^* - r}{r} S(\phi). \quad (5.4)$$

From (4.14) and (5.4), the proof follows. \square

From Proposition 5.1 and Proposition 5.2, we have the following corollary.

Corollary 5.3 *On a Kähler foliation (\mathcal{F}, J) , if ϕ is a transverse conformal Killing r -form, then*

$$F(J\phi) = 0.$$

Let ϕ be a transverse conformal Killing r -form. Then it is trivial that from Lemma 4.9,

$$d_B^c \phi = \frac{r^* + 1}{rr^* - r - 2} d_B J\phi - \frac{2(r + 1)}{rr^* - r - 2} \delta_T L\phi, \quad (5.5)$$

$$\delta_T^c \phi = \frac{r + 1}{rr^* - r^* - 2} \delta_T J\phi + \frac{2(r^* + 1)}{rr^* - r^* - 2} d_B \Lambda\phi. \quad (5.6)$$

Hence we have the following lemma.

Lemma 5.4 *Let ϕ be a transverse conformal Killing r -form. Then*

$$\delta_B d_B^c \phi = \frac{r^* + 1}{rr^* - r - 2} \delta_B d_B J\phi + \frac{2(r + 1)}{rr^* - r - 2} \delta_B i(\kappa_B^\sharp) L\phi, \quad (5.7)$$

$$d_B \delta_T^c \phi = \frac{r + 1}{rr^* - r^* - 2} d_B \delta_T J\phi, \quad (5.8)$$

$$\delta_B \delta_T^c \phi = \frac{2(r^* + 1)}{rr^* - r^* - 2} \delta_B d_B \Lambda\phi - \frac{r + 1}{rr^* - r^* - 2} \delta_B i(\kappa_B^\sharp) J\phi. \quad (5.9)$$

Lemma 5.5 *Let ϕ be a transverse conformal Killing r -form. Then*

$$a_1 \delta_T d_B J\phi + a_2 d_B \delta_T J\phi + a_3 \{i(\kappa_B^\sharp) \delta_B \phi + \delta_B i(\kappa_B^\sharp) \phi\} = 0,$$

where $a_1 = \frac{r^2 r^* - r - rr^* - r^*}{(r+1)(rr^* - r - 2)}$, $a_2 = \frac{rr^{*2} - r^* - rr^* - r}{(r^*+1)(rr^* - r^* - 2)}$ and $a_3 = \frac{2(r - r^*)}{(r^*+1)(rr^* - r - 2)}$.

Proof. From Proposition 3.2, Lemma 4.9, Corollary 5.3 and Lemma 5.4, the proof follows. \square

Theorem 5.6 *Let (M, g_M, \mathcal{F}, J) be a closed, connected Riemannian manifold with a Kähler foliation of codimension $q = 2m$. If ϕ is a transverse conformal Killing m -form, then $J\phi$ is parallel.*

Proof. Since $\phi \in \Omega_B^m(\mathcal{F})$ be a transverse conformal Killing m -form, i.e., $\phi \in \Omega_B^m(\mathcal{F})$, $a_1 = a_2$ and $a_3 = 0$ in Lemma 5.5. Hence

$$d_B \delta_T J\phi + \delta_T d_B J\phi = 0.$$

Therefore, we have

$$\Delta_B J\phi = \theta(\kappa_B^\sharp) J\phi. \quad (5.10)$$

Hence, by the generalized Weitzenböck formula (Corollary 2.3),

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp) |J\phi|^2 = -|\nabla_{\text{tr}} J\phi|^2 \leq 0. \quad (5.11)$$

From the generalized maximum principle (Theorem 2.4), $|J\phi|$ is constant. Again, from (5.11), we have

$$\nabla_{\text{tr}} J\phi = 0, \quad (5.12)$$

which implies that $J\phi \in \Omega_B^m(\mathcal{F})$ is parallel. \square

On the other hand, for any basic r -form ϕ , Lemma 4.9 implies that

$$J\Lambda d_B \delta_B \phi = d_B \delta_B J\Lambda \phi + d_B^c \delta_B \Lambda \phi + d_B \delta_B^c \Lambda \phi + J\delta_T^c \delta_B \phi, \quad (5.13)$$

$$J\Lambda \delta_B d_B \phi = \delta_B d_B J\Lambda \phi + \delta_B d_B^c \Lambda \phi + \delta_B^c d_B \Lambda \phi + J\delta_B \delta_T^c \phi. \quad (5.14)$$

Hence we have the following lemma.

Lemma 5.7 *Let ϕ be a transverse conformal Killing r -form. Then*

$$J\Lambda d_B \delta_B \phi = d_B \delta_B J\Lambda \phi + d_B \delta_B^c \Lambda \phi + d_B^c \delta_B \Lambda \phi - \frac{2(r^* + 1)}{r^*(r + 1)} J\Lambda \delta_B d_B \phi \quad (5.15)$$

$$- Je(J\kappa_B^\sharp)\phi + \frac{1}{r^*} J\delta_B i(\kappa_B^\sharp) J\phi,$$

$$J\Lambda \delta_B d_B \phi = \delta_B d_B J\Lambda \phi + \delta_B d_B^c \Lambda \phi + \delta_B^c d_B \Lambda \phi + \frac{2(r^* + 1)}{r^*(r + 1)} J\Lambda \delta_B d_B \phi \quad (5.16)$$

$$- \frac{1}{r^*} J\delta_B i(\kappa_B^\sharp) J\phi.$$

Proof. Let ϕ be a transverse conformal Killing r -form. Since $\delta_B^c \delta_B \phi = -\delta_B \delta_T^c \phi - E(J\kappa_B^\sharp)\phi$, from Lemma 5.4 we have

$$\begin{aligned} J\delta_T^c \delta_B \phi &= -\frac{2(r^* + 1)}{rr^* - r^* - 2} J\delta_B d_B \Lambda \phi + \frac{r + 1}{rr^* - r^* - 2} J\delta_B i(\kappa_B^\sharp) J\phi - Je(J\kappa_B^\sharp)\phi \\ &= -\frac{2(r^* + 1)}{rr^* - r^* - 2} J\Lambda \delta_B d_B \phi + \frac{2(r^* + 1)}{rr^* - r^* - 2} J\delta_B \delta_T^c \phi - Je(J\kappa_B^\sharp)\phi \\ &\quad + \frac{r + 1}{rr^* - r^* - 2} J\delta_B i(\kappa_B^\sharp) J\phi. \end{aligned}$$

Hence we have

$$J\delta_T^c\delta_B\phi = -\frac{2(r^*+1)}{r^*(r+1)}J\Lambda\delta_B d_B\phi - Je(J\kappa_B^\#)\phi + \frac{1}{r^*}J\delta_B i(\kappa_B^\#)J\phi.$$

From (5.13), the proof of (5.15) follows. The proof of (5.16) is similar from (5.14). \square

Lemma 5.8 *Let (\mathcal{F}, J) be a minimal Kähler foliation. Then for a transverse conformal Killing r ($2 \leq r \leq q-2$)-form ϕ ,*

$$\delta_B^c d_B \Lambda \phi = -\frac{r+1}{(r-1)(r^*+1)}\{J\Lambda d_B \delta_B \phi + \delta_B d_B^c \Lambda \phi\}, \quad (5.17)$$

$$\delta_B d_B^c \Lambda \phi = \frac{r^*+1}{(r+1)(r^*-1)}\{J\Lambda \delta_B d_B \phi - \delta_B^c d_B \Lambda \phi\}. \quad (5.18)$$

Proof. From Lemma 4.9 and Lemma 5.4, we have

$$\begin{aligned} \delta_B^c d_B \Lambda \phi &= \Lambda \delta_B^c d_B \phi - \delta_B^c \delta_T^c \phi \\ &= -\lambda\{J\Lambda d_B \delta_B \phi - \Lambda d_B^c \delta_B \phi - \Lambda d_B \delta_B^c \phi\} - \delta_B^c \delta_T^c \phi, \end{aligned}$$

where $\lambda = \frac{r+1}{rr^*-r^*-2}$. Since

$$\begin{aligned} \Lambda d_B \delta_B^c \phi &= -\delta_B^c d_B \Lambda \phi + \delta_T^c \delta_B^c \phi + \theta(J\kappa_B^\#)\Lambda \phi, \\ \Lambda d_B^c \delta_B \phi &= -\delta_B d_B^c \Lambda \phi + \theta(J\kappa_B^\#)^t \Lambda \phi - \delta_T \delta_B \phi. \end{aligned}$$

we have

$$\begin{aligned} (1+\lambda)\delta_B^c d_B \Lambda \phi &= -\lambda\{J\Lambda d_B \delta_B \phi - \Lambda d_B^c \delta_B \phi\} + \lambda\{\delta_T^c \delta_B^c \phi + \theta(J\kappa_B^\#)\phi\} - \delta_B^c \delta_T^c \phi \\ &= -\lambda\{J\Lambda d_B \delta_B \phi + \delta_B d_B^c \Lambda \phi\} + \lambda\{\theta(J\kappa_B^\#)^t \phi + \theta(J\kappa_B^\#)\phi\} \\ &\quad + \lambda\{\delta_T^c \delta_B^c \phi - \delta_T \delta_B \phi\} - \delta_B^c \delta_T^c \phi. \end{aligned}$$

Since \mathcal{F} is minimal, $\delta_T^c \delta_B^c \phi = \delta_B^c \delta_B^c \phi = 0$ and $\delta_T \delta_B \phi = 0$, the above equation proves (5.17). Similarly, from Lemma 5.4, (5.18) is proved. \square

Now, we put

$$x = J\Lambda(d_B \delta_B \phi), \quad y = J\Lambda(\delta_B d_B \phi), \quad \alpha = \delta_B^c d_B \Lambda \phi, \quad (5.19)$$

$$\beta = \delta_B d_B^c \Lambda \phi, \quad a = d_B \delta_B J\Lambda \phi, \quad b = \delta_B d_B J\Lambda \phi. \quad (5.20)$$

From now on, we assume that \mathcal{F} is minimal. From Lemma 5.7, we have

$$x = a - \alpha - \beta - \frac{2(r^* + 1)}{r^*(r + 1)}y, \quad (5.21)$$

$$y = b + \alpha + \beta + \frac{2(r^* + 1)}{r^*(r + 1)}y. \quad (5.22)$$

Hence from (5.21) and (5.22), we have

$$y = \lambda_1(b + \alpha + \beta), \quad \lambda_1 = \frac{r^*(r + 1)}{r^*(r - 1) - 2}, \quad (5.23)$$

$$x = a + (1 - \lambda_1)b - \lambda_1(\alpha + \beta). \quad (5.24)$$

From Lemma 5.8, we have

$$\alpha = -\lambda_2(x + \beta), \quad \lambda_2 = \frac{r + 1}{(r^* + 1)(r - 1)}, \quad (5.25)$$

$$\beta = \lambda_3(y - \alpha), \quad \lambda_3 = \frac{r^* + 1}{(r + 1)(r^* - 1)}. \quad (5.26)$$

From (5.23), (5.24), (5.25) and (5.26), we have

$$\lambda_1 \lambda_3 b = (1 - \lambda_1 \lambda_3)\beta + \lambda_3(1 - \lambda_1)\alpha, \quad (5.27)$$

$$\lambda_2 a + \lambda_2(1 - \lambda_1)b = (\lambda_1 \lambda_2 - 1)\alpha + \lambda_2(\lambda_1 - 1)\beta. \quad (5.28)$$

Hence we have the following theorem.

Theorem 5.9 *Let (M, g_M, J, \mathcal{F}) be a closed Riemannian manifold with a minimal Kähler foliation \mathcal{F} of codimension $q = 2m$ and a bundle-like metric g_M . Then for any transverse conformal Killing r ($2 \leq r \leq q - 2$)-form ϕ , $J\Lambda\phi$ is basic-harmonic.*

Proof. From Lemma 4.9, $d_B \delta_B^c + \delta_B^c d_B = \theta(J\kappa_B^\sharp)\phi$. Hence we have

$$\int_M \langle b, \alpha \rangle \mu_M = \int_M \langle d_B J\Lambda\phi, \theta(J\kappa_B^\sharp) d_B \Lambda\phi \rangle \mu_M.$$

Since \mathcal{F} is minimal, we have

$$\int_M \langle b, \alpha \rangle \mu_M = 0.$$

Similary. we have

$$\int_M \langle \beta, \alpha \rangle \mu_M = 0.$$

Hence from (5.27), we have

$$\lambda_3(1 - \lambda_1) \int_M |\alpha|^2 \mu_M = 0.$$

Since $\lambda_3 \neq 0$ and $\lambda_1 \neq 1$, $\alpha = 0$. Similary, from Lemma 4.9,

$$\int_M \langle a, b \rangle \mu_M = \int_M \langle a, \beta \rangle \mu_M = 0,$$

which implies $a = 0$ from (5.28). Therefore, from (5.27) and (5.28), since $\lambda_2(1 - \lambda_1) \neq 0$, we have

$$\lambda_1 \lambda_3 b = (1 - \lambda_1 \lambda_3) \beta, \quad b = -\beta.$$

Hence $b = \beta = 0$. Therefore, $x = y = 0$. So from Corollary 4.12, $\Delta_B J\Lambda\phi = J\Lambda\Delta_B\phi = 0$. That is, $J\Lambda\phi$ is basic-harmonic. \square

Corollary 5.10 *Let (M, g_M, J, \mathcal{F}) be as in Theorem 5.9. Then for a transverse conformal Killing r ($2 \leq r \leq q - 2$)-form ϕ , $J\Lambda\phi$ is parallel.*

Proof. Let ϕ be a transverse conformal Killing form. Then from Corollary 5.3, $F(J\Lambda\phi) = \Lambda F(J\phi) = 0$. Since \mathcal{F} is minimal, from Theorem 5.9, $\Delta_B(J\Lambda\phi) = 0$. Hence from the generalized Weitzenböck formula (Theorem 2.2), we have

$$\nabla_{\text{tr}}^* \nabla_{\text{tr}} \Lambda J\phi = 0,$$

which means that $J\Lambda\phi$ is parallel. \square

Theorem 5.11 *Let (M, g_M, J, \mathcal{F}) be as in Theorem 5.9. Then for a transversal conformal Killing r ($2 \leq r \leq q - 2$)-form, $J\phi$ is parallel.*

Proof. Let ϕ be a transverse conformal Killing r -form. Then $\bar{*}\phi$ is also a transverse conformal Killing $(q-r)$ -form [6]. Hence by Corollary 5.10, $J\Lambda\bar{*}\phi$ is parallel. Since $[\nabla_{\text{tr}}, \bar{*}] = 0$, $[J, \bar{*}] = 0$ and $L\bar{*} = \bar{*}\Lambda$, $\bar{*}J\Lambda\bar{*}\phi = \pm L J\phi$ is parallel. Note that $(m-r)J\phi = [\Lambda, L]J\phi$. Since $[L, \nabla_{\text{tr}}] = [\Lambda, \nabla_{\text{tr}}] = 0$, if $r \neq m$, then $J\phi$ is parallel. For $r = m$, from Theorem 5.6, $J\phi$ is parallel. So the proof is completed. \square

Question. When \mathcal{F} is not minimal, is Theorem 5.11 true?

Remark. For the point foliation, Theorem 5.11 can be found in [13].

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